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THREE-DIMENSIONAL IMPLICIT LAMBDA METHODS

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## THREE-DIMENSIONAL IMPLICIT LAMBDA METHODS\*

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### ABSTRACT

This paper derives the three-dimensional lambda-formulation equations for a general orthogonal curvilinear coordinate system and provides various block-explicit and block-implicit methods for solving them, numerically. Three model problems, characterized by subsonic, supersonic and transonic flow conditions, are used to assess the reliability and compare the efficiency of the proposed methods.

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## INTRODUCTION

Among the many theoretical models employed in the numerical simulation of compressible inviscid flows the so-called lambda-formulation has received considerable interest (see, e.g., [1-8]): the time-dependent Euler equations are recast into compatibility conditions of bicharacteristic variables along the corresponding bicharacteristic lines and discretized using windward differences, in order to account for the direction of wave propagation phenomena, correctly. Such an approach has many nice properties: it provides very accurate numerical results, even with rather coarse meshes (see, e.g., [2], [3], [6]); it requires only the physical boundary conditions, so that there is no need for any additional numerical boundary treatments, which are frequently the cause of numerical instability [9]; it handles in a most automatic and physically-sound way mixed supersonic-subsonic flow fields; and finally, it has a well documented, although controversial, capability of capturing shocks without any additional dissipation [2-6]. For these reasons, in spite of the fact that the "captured shocks" are only isentropic approximations to correct weak solutions of the Euler equations and do not correctly move within the flow field - unless properly fitted [4], the lambda-formulation is considered to be a very useful and reliable tool for predicting compressible flow fields and, therefore, very worthy of further studies and improvements; and in fact, in the last two years, for the cases of quasi-one dimensional and two dimensional flows, the development of various kinds of implicit integration schemes [5-8] has removed the only major limitation of previous lambda methods, namely, the CFL stability restriction associated with their explicit integration procedures.

It now appears very timely and worthwhile to develop efficient numerical methods, based on the lambda-formulation, for three-dimensional flows, as done

in the present paper: the "most appropriate" three-dimensional lambda-formulation equations are first derived for the case of a general orthogonal curvilinear coordinate system; the governing equations are then discretized and linearized in time using a delta approach and various block-explicit as well as block-implicit numerical techniques are proposed to solve the resulting discrete equations approximately at every time step; all of the proposed methods are finally applied to solve three model problems, characterized by subsonic, supersonic and transonic flow conditions, respectively, in order to assess their reliability and efficiency.

### THREE-DIMENSIONAL LAMBDA-FORMULATION EQUATIONS

The nondimensional continuity and momentum (Euler) equations for the homentropic flow of a perfect gas are given in vector form as [3,5]

$$\delta(a_t + \underline{V} \cdot \underline{\nabla} a) + a \underline{\nabla} \cdot \underline{V} = 0 \quad (1)$$

$$\underline{V}_t + (\underline{V} \cdot \underline{\nabla}) \underline{V} + \delta a \underline{\nabla} a = 0, \quad (2)$$

where  $a$  is the speed of sound,  $\underline{V}$  is the velocity vector,  $\underline{\nabla}$  is the gradient operator,  $t$  is the time, subscripts indicate partial derivatives and  $\delta = 2/(\gamma-1)$ ,  $\gamma$  being the specific heats ratio.

In a general orthogonal curvilinear coordinate system we have:

$$\underline{V} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \quad (3)$$

$$\underline{\nabla} = \underline{e}_1 \frac{\partial}{\partial s_1} + \underline{e}_2 \frac{\partial}{\partial s_2} + \underline{e}_3 \frac{\partial}{\partial s_3} = \frac{\underline{e}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\underline{e}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\underline{e}_3}{h_3} \frac{\partial}{\partial q_3} \quad (4)$$

$$\underline{v} \cdot \underline{v} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial q_1} (h_2 h_3 v_1) + \frac{\partial}{\partial q_2} (h_1 h_3 v_2) + \frac{\partial}{\partial q_3} (h_1 h_2 v_3) \right], \quad (5)$$

where  $\underline{e}_i (i = 1, 2, 3)$  are the unit vectors of the (right-handed) orthogonal curvilinear coordinate system,  $q_i$  and  $h_i$  are the corresponding coordinates and scale factors,  $ds_i = h_i dq_i$  are the elementary arc lengths along the coordinate lines, see [10], and  $v_i$  are the components of  $\underline{v}$ . Equations (1) and (2) can be written in the general orthogonal curvilinear coordinate system by means of eqs. (3-5) and some lengthy but straightforward algebra, the only difficulty being the evaluation of the derivatives of the unit vectors  $\underline{e}_i$  with respect to the coordinates  $q_i$ . These expressions are therefore given here as:

$$\frac{\partial \underline{e}_i}{\partial q_j} = \frac{\underline{e}_j}{h_j} \frac{\partial h_i}{\partial q_j} \quad (i \neq j) \quad (6)$$

$$\frac{\partial \underline{e}_i}{\partial q_j} = -\frac{\underline{e}_j}{h_j} \frac{\partial h_i}{\partial q_j} - \frac{\underline{e}_k}{h_k} \frac{\partial h_i}{\partial q_k} \quad (i \neq j \neq k). \quad (7)$$

The six lambda compatibility equations can now be obtained by summing and subtracting from the continuity eqn. (1) each component of the momentum eqn. (2), to give:

$$C_t + (v_1 + a) \frac{\partial C}{\partial s_1} + v_2 \frac{\partial C}{\partial s_2} + v_3 \frac{\partial C}{\partial s_3} = -\zeta - \alpha_1 + \beta_1 - \gamma_1 \quad (8a)$$

$$D_t + (v_1 - a) \frac{\partial D}{\partial s_1} + v_2 \frac{\partial D}{\partial s_2} + v_3 \frac{\partial D}{\partial s_3} = \zeta - \alpha_1 + \beta_1 + \gamma_1 \quad (8b)$$

$$E_t + v_1 \frac{\partial E}{\partial s_1} + (v_2 + a) \frac{\partial E}{\partial s_2} + v_3 \frac{\partial E}{\partial s_3} = -\zeta - \alpha_2 + \beta_2 - \gamma_2 \quad (8c)$$

$$F_t + v_1 \frac{\partial F}{\partial s_1} + (v_2 - a) \frac{\partial F}{\partial s_2} + v_3 \frac{\partial F}{\partial s_3} = \zeta - \alpha_2 + \beta_2 + \gamma_2 \quad (8d)$$

$$G_t + v_1 \frac{\partial G}{\partial s_1} + v_2 \frac{\partial G}{\partial s_2} + (v_3 + a) \frac{\partial G}{\partial s_3} = -\zeta - \alpha_3 + \beta_3 - \gamma_3 \quad (8e)$$

$$H_t + v_1 \frac{\partial H}{\partial s_1} + v_2 \frac{\partial H}{\partial s_2} + (v_3 - a) \frac{\partial H}{\partial s_3} = \zeta - \alpha_3 + \beta_3 + \gamma_3 \quad (8f)$$

where C, D, E, F, G and H are the six bicharacteristic variables given as

$$C = v_1 + \delta a; \quad D = v_1 - \delta a; \quad E = v_2 + \delta a \quad (9a,b,c)$$

$$F = v_2 - \delta a; \quad G = v_3 + \delta a; \quad H = v_3 - \delta a \quad (9d,e,f)$$

$$\zeta = \frac{a}{h_1 h_2 h_3} \left[ v_1 \frac{\partial}{\partial q_1} (h_2 h_3) + v_2 \frac{\partial}{\partial q_2} (h_1 h_3) + v_3 \frac{\partial}{\partial q_3} (h_1 h_2) \right] \quad (10)$$

$$\alpha_1 = \frac{v_1}{h_1} \left( \frac{v_2}{h_2} \frac{\partial h_1}{\partial q_2} + \frac{v_3}{h_3} \frac{\partial h_1}{\partial q_3} \right) \quad (11a)$$

$$\beta_1 = \frac{v_2^2}{h_1 h_2} \frac{\partial h_2}{\partial q_1} + \frac{v_3^2}{h_1 h_3} \frac{\partial h_3}{\partial q_1} \quad (11b)$$

$$\gamma_1 = a \left( \frac{1}{h_2} \frac{\partial v_2}{\partial q_2} + \frac{1}{h_3} \frac{\partial v_3}{\partial q_3} \right) \quad (11c)$$

and  $\alpha_2, \beta_2, \dots, \gamma_3$  have very similar expressions, which can be obtained by appropriate subscripts rotation and are thus omitted for the sake of conciseness.

Equations (8) are the compatibility conditions of the bicharacteristic variables along their bicharacteristic lines (in the four-dimensional  $q_1, q_2, q_3, t$  space) obtained by the intersection of the bicharacteristic conoid (associated with a point P) with the  $q_1-t, q_2-t$  and  $q_3-t$  surfaces passing through its vertex P (the left-hand sides of eqns. (8) clearly being total derivatives along such lines). Therefore, they could be integrated by means



of any numerical method using windward differences according to the direction of wave propagation along the bicharacteristic lines, thus providing a three-dimensional lambda scheme. However, like in the two-dimensional case [6], there are two major difficulties associated with solving eqns. (8) numerically. First, the six bicharacteristic variables are not independent, insofar as their very definitions, eqns. (9), imply that:

$$v_1 = \frac{C + D}{2}, \quad v_2 = \frac{E + F}{2}, \quad v_3 = \frac{G + H}{2} \quad (12,a,b,c)$$

$$\delta a = C - D = E - F = G - H \quad (12,d,e,f)$$

so that

$$F = -C + D + E \quad (13)$$

$$H = -C + D + G. \quad (14)$$

Therefore, any numerical solution obtained by integrating eqns. (8), directly, would lead to a "nonuniqueness" in the value of the speed of sound  $a$ . Furthermore, the right-hand sides of eqns. (8), namely the  $\gamma_i$  coefficients, contain spatial derivatives of the velocity components, which are not associated with the convection of physical disturbances and are therefore likely to reduce the accuracy of the spatial discretization, if not the stability of the integration process. For these reasons, as in [5,6] for the two-dimensional case, the following equivalent system is obtained by taking a complete set of appropriate linear combinations of eqns (8):

$$\begin{aligned}
C_t + D_t + (v_1+a) \frac{\partial C}{\partial s_1} + (v_1-a) \frac{\partial D}{\partial s_1} + v_2 \frac{\partial}{\partial s_2} (C + D) \\
+ v_3 \frac{\partial}{\partial s_3} (C + D) = -2\alpha_1 + 2\beta_1
\end{aligned} \tag{15}$$

$$\begin{aligned}
E_t + F_t + v_1 \frac{\partial}{\partial s_1} (E + F) + (v_2+a) \frac{\partial E}{\partial s_2} + (v_2-a) \frac{\partial F}{\partial s_2} \\
+ v_3 \frac{\partial}{\partial s_3} (E + F) = -2\alpha_2 + 2\beta_2
\end{aligned} \tag{16}$$

$$\begin{aligned}
G_t + H_t + v_1 \frac{\partial}{\partial s_1} (G + H) + v_2 \frac{\partial}{\partial s_2} (G + H) + (v_3+a) \frac{\partial G}{\partial s_3} \\
+ (v_3-a) \frac{\partial H}{\partial s_3} = -2\alpha_3 + 2\beta_3
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{1}{3} \{ C_t - D_t + E_t - F_t + G_t - H_t \} + (v_1+a) \frac{\partial C}{\partial s_1} - (v_1-a) \frac{\partial D}{\partial s_1} \\
+ (v_2+a) \frac{\partial E}{\partial s_2} - (v_2-a) \frac{\partial F}{\partial s_2} + (v_3+a) \frac{\partial G}{\partial s_3} - (v_3-a) \frac{\partial H}{\partial s_3} = -2\zeta
\end{aligned} \tag{18}$$

$$C_t - D_t - E_t + F_t = 0 \tag{19}$$

$$C_t - D_t - G_t + H_t = 0. \tag{20}$$

It is noteworthy that eqns. (15-17) are simply the three components of the momentum eqn. (2), expressed as the sums of two compatibility conditions of two bicharacteristic variables along their bicharacteristic lines, whereas eqns. (18-20) all coincide with the continuity eqn. (1). Also, eqns. (19) and

(20), after a straightforward integration with respect to time, identically reproduce eqns. (13) and (14), so that they effectively reduce the number of dependent variables from six to four and any numerical integration of eqns. (13-18) will guarantee a unique solution for the physical variable  $a$ . Finally, the right-hand sides of eqns. (15-18) are seen to contain only source-like terms which do not involve spatial derivatives of the dependent variables and, therefore, are not likely to deteriorate the accuracy of any numerical method using windward differences for the "total" derivatives of the bicharacteristics variables.

For these reasons eqns. (13-18) are considered the "most appropriate" three-dimensional lambda-formulation equations for a general orthogonal coordinate system and will be the basis for all of the numerical methods proposed in this study.

## NUMERICAL METHODS

The governing equations (13-18) are discretized and linearized in time using the delta form [11,5,6] to give

$$\begin{aligned} \frac{\Delta C}{\Delta t} + \frac{\Delta D}{\Delta t} + (u+a)^n \Delta C_x + (u-a)^n \Delta D_x + v^n (\Delta C_y + \Delta D_y) + w^n (\Delta C_z + \Delta D_z) \\ = - (u+a)^n C_x^n - (u-a)^n D_x^n - v^n (C_y^n + D_y^n) - w^n (C_z^n + D_z^n) \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\Delta E}{\Delta t} + \frac{\Delta F}{\Delta t} + u^n (\Delta E_x + \Delta F_x) + (v+a)^n \Delta E_y + (v-a)^n \Delta F_y + w^n (\Delta E_z + \Delta F_z) \\ = -u^n (E_x^n + F_x^n) - (v+a)^n E_y^n - (v-a)^n F_y^n - w^n (E_z^n + F_z^n) \end{aligned} \quad (22)$$

$$\begin{aligned}
& \frac{\Delta G}{\Delta t} + \frac{\Delta H}{\Delta t} + u^n(\Delta G_x + \Delta H_x) + v^n(\Delta G_y + \Delta H_y) + (w+a)^n \Delta G_z + (w-a)^n \Delta H_z \\
& = -u^n(G_x + H_x)^n - v^n(G_y + H_y)^n - (w+a)^n G_z^n - (w-a)^n H_z^n
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \frac{1}{3} \left\{ \frac{\Delta C}{\Delta t} - \frac{\Delta D}{\Delta t} + \frac{\Delta E}{\Delta t} - \frac{\Delta F}{\Delta t} + \frac{\Delta G}{\Delta t} - \frac{\Delta H}{\Delta t} \right\} + (u+a)^n \Delta C_x - (u-a)^n \Delta D_x + (v+a)^n \Delta E_y \\
& - (v-a)^n \Delta F_y + (w+a)^n \Delta G_z - (w-a)^n \Delta H_z = -(u+a)^n C_x^n + (u-a)^n D_x^n \\
& - (v+a)^n E_y^n + (v-a)^n F_y^n - (w+a)^n G_z^n + (w-a)^n H_z^n
\end{aligned} \tag{24}$$

$$\Delta F = -\Delta C + \Delta D + \Delta E \tag{25}$$

$$\Delta H = -\Delta C + \Delta F + \Delta G, \tag{26}$$

where Cartesian coordinates have been used for simplicity,  $\Delta t$  is the time step,  $\Delta C = C^{n+1} - C^n$  (the superscripts  $n+1$  and  $n$  indicating the new and old time levels  $t^{n+1} = t^n + \Delta t$  and  $t^n$ ) etc. Equations (21-26) constitute a first-order-accurate implicit time discretization of the corresponding differential problem; eqns. (21-24) are then discretized in space, using windward differences to properly account for the direction of wave propagation, and the  $\Delta F$  and  $\Delta H$  unknowns are eliminated in favor of  $\Delta C$ ,  $\Delta D$ ,  $\Delta E$  and  $\Delta G$  by means of eqns. (25) and (26) to produce, together with appropriate boundary conditions, a large  $4 \times 4$  block-sparse linear system of the type

$$A f = b. \tag{27}$$

For the case of a cubic integration domain having  $N$  gridpoints in every spatial direction,  $A$  is a square matrix of order  $N^3$  having only seven nonzero diagonals of  $4 \times 4$ -matrix-elements,  $f$  is the unknown vector having  $N^3$  four-element-vector-components and  $b$  is the known coefficient vector. It is noteworthy that in previous works [5,6] a second-order-accurate time linearization was employed. However, due to the use of a backward Euler time discretization, eqn. (27) is only first-order-accurate in time anyway. Moreover, the present linearization, coupled with windward difference approximations for the left-hand sides of eqns. (21-24), leads to a diagonally dominant matrix  $A$  and has been verified to increase the stability of all the implicit methods later proposed in this study. It is also noteworthy that second-order-accurate, three-points windward differences can be used to approximate the right-hand sides of eqns. (21-24) so that, if the flow reaches a steady state, the final solution is second-order-accurate [5,6].

The main reason to employ an implicit method is to remove the CFL stability restriction, thus improving the efficiency of the calculations. However, a direct solution of problem (27), even if feasible, is certainly impractical. Therefore, the matrix  $A$  will be replaced by a matrix  $B$  which is easily invertible and is a first-order-accurate approximation (in time) of  $A$ .

#### A Block-Explicit Method

The simplest first-order-accurate approximation to  $A$  can be obtained by dropping all but the time-derivative terms in the left-hand sides of eqns. (21-24). The resulting matrix  $B$  is diagonal and a simple  $4 \times 4$  linear system needs to be solved at every gridpoint to provide the local  $\Delta C$ ,  $\Delta D$ ,  $\Delta E$ , and  $\Delta G$  values. Furthermore, eqns. (21-24) can be rearranged to give

$$\frac{2\Delta C}{\Delta t} = \text{RHS}(21) + \text{RHS}(24) \quad (28)$$

$$\frac{\Delta C}{\Delta t} + \frac{\Delta D}{\Delta t} = \text{RHS}(21) \quad (29)$$

$$-\frac{\Delta C}{\Delta t} + \frac{\Delta D}{\Delta t} + \frac{2\Delta E}{\Delta t} = \text{RHS}(22) \quad (30)$$

$$-\frac{\Delta C}{\Delta t} + \frac{\Delta D}{\Delta t} + \frac{2\Delta G}{\Delta t} = \text{RHS}(23) \quad (31)$$

(where  $\text{RHS}(21)$  is a shorthand notation for the right-hand side of eqn. (21), etc.) so that every element of  $\mathbf{B}$  is a lower triangular matrix which can be inverted directly. The present BE method has been developed mainly for assessing the efficiency of various implicit methods; however, due to its extreme coding simplicity, it could very well be a useful tool by itself, especially if implemented on a vector computer.

#### A Block-Alternating-Direction-Implicit (BADI) Method

An ADI technique has been developed, which is the direct extension to three-dimensional problems of the method of Refs. [5] and [6]. A three-sweep ADI process is used to solve problem (27) approximately. At the first sweep the  $t$  and  $x$  derivatives in the left-hand sides of eqns. (21-24) are evaluated implicitly, whereas the  $y$  and  $z$  derivatives are evaluated explicitly. At the second and third sweeps the  $t$  and  $y$  and the  $t$  and  $z$  derivatives are evaluated implicitly so that  $\mathbf{A}$  is approximated by the product of three  $4 \times 4$  block-tridiagonal matrices. In practice, at every sweep of the BADI method a  $4 \times 4$  block-tridiagonal system of order  $N$  has to be solved along each line of the computational grid, so that  $3N^2$  such systems need to be solved at every time step (i.e., to solve eqn. (27))

approximately). With respect to two-dimensional flow problems, the present ADI method is less competitive as compared to a standard explicit method, for two reasons: the block size of the tridiagonal systems increases from three to four and, more importantly, the number of tridiagonal systems to be solved at every time step grows from  $2N$  to  $3N^2$ . Actually, for the simple problems later considered in this study the computer time per step for an  $11^3$  mesh was found to be about 30 times greater than that required by the RE method. More efficient implicit methods need therefore to be devised for the three-dimensional lambda-formulation equations.

#### A Block-Line-Gauss-Seidel (BLGS) Method

Classical relaxation methods have been recently employed with considerable success in connection with "upwind schemes" for the one- and two-dimensional Euler equations [7,8,12]. Here an obvious choice, leading to a reduction of the computer time per step to about one third, is to employ a single step  $4 \times 4$  block-line-Gauss-Seidel method: all of the time and  $x$  derivatives in the left-hand sides of eqns. (21-24) are evaluated implicitly together with the diagonal contributions of the  $y$  and  $z$  derivatives, so that only  $N^2$   $4 \times 4$  block-tridiagonal systems (of order  $N$ ) have to be solved at every time level. By accounting for the previously evaluated nontridiagonal entries explicitly, the matrix  $A$  is effectively replaced by its three main diagonals plus its two additional nonzero lower diagonals. Furthermore, the ordering of the solution process is changed at every time step so as to account for the two additional nonzero upper or lower diagonals, alternately.

### A Block-Point-Gauss-Seidel (BPGS) Method

By taking to its extreme the logic behind the previous method, an obvious choice presents itself; that is, to replace the matrix  $A$  with its lower or upper triangular part. In eqns. (21-24) the diagonal contributions are accounted for implicitly and the previously evaluated off-diagonal contributions are brought to the right-hand sides of the equations and accounted for explicitly. At every gridpoint location a  $4 \times 4$  linear system needs to be solved as in the BE method; however, due to its variable coefficients, the local  $4 \times 4$  matrix cannot be triangularized and a complete Gauss-Jordan elimination procedure, using diagonal pivot strategy, has been employed (here as well as to solve the local linear systems within a general block-tridiagonal inversion routine in all of the present implicit methods).

### A Simplified-Line-Gauss-Seidel (SLGS) Method

From their very definitions (eqns. (9)) as well as from their compatibility conditions (eqns. (8)) it appears that the waves associated with the bicharacteristic variables  $C$  and  $D$  mainly propagate in the  $x$  direction, whereas the  $E$  and  $F$  waves and the  $G$  and  $H$  waves mainly propagate in the  $y$  and  $z$  directions, respectively. Therefore, it would seem appropriate to devise a numerical method exploiting such a property of the compatibility eqns. (8), as done in [7,8] for the case of one- and two-dimensional flows. However, whereas Moretti [7,8] integrates the compatibility conditions directly, here eqns. (13-18) are preferred for the two reasons previously discussed. In conclusion, the following simplified line-Gauss-Seidel method is proposed here: Equations (21) and (24) are solved coupled together for the  $\Delta C$  and  $\Delta D$  variables by means of a line-Gauss-Seidel method, implicit in the  $x$  direction, so that a  $2 \times 2$  block-tridiagonal system of



order  $N$  has to be solved at every  $y_j$  and  $z_k$  gridpoint location. Equations (22) and (23) are then solved by means of line-Gauss-Seidel methods implicit in the  $y$  and  $z$  direction, respectively, so that  $2N^2$  additional scalar tridiagonal systems need to be solved. Obviously, equations (25) and (26) are used to eliminate  $\Delta F$  and  $\Delta H$  from eqns. (21-24) and all of the  $\Delta C, \dots, \Delta H$  terms already evaluated at any level of the computation process are accounted for in the right-hand sides of the equations. Furthermore, since the pressure eqn. (24) does not have a main direction of propagation, it is coupled to eqn. (22), to evaluate  $\Delta E$  and  $\Delta F$  implicitly in the  $y$  direction, and to eqn. (23), to evaluate  $\Delta G$  and  $\Delta H$  implicitly in the  $z$  direction, at successive time steps.

It is noteworthy that, in general, the matrix  $A$  contains all the boundary conditions, which are therefore accounted for with the level of implicitness typical of every single method. However, for simplicity, in all of the present applications the exact solution of the continuum problem has been enforced at all boundaries to provide homogeneous boundary conditions for all of the incremental bicharacteristic variables. More general boundary conditions can be implemented as suggested in [6] and are not expected to cause any difficulty.

## RESULTS

In order to test the proposed methods, a simple steady one-dimensional spherical source flow of air ( $\gamma = 1.4$ ) has been considered; for such a flow field the continuity and energy equations are given as

$$a^5 v_r r^2 = c_1 \quad (32)$$

$$0.2 v_r^2 + a^2 = c_2 \quad (33)$$

$v_r$  being the radial velocity component and  $r$  the radial distance from the origin. All of the calculations have been performed using a Cartesian coordinate system inside the unit cube such that:  $2 \leq x \leq 3$ ;  $-.5 \leq y \leq .5$ ;  $-.5 \leq z \leq .5$ . Three flow conditions have been considered: the subsonic flow corresponding to  $c_1 = 3.2$  and  $c_2 = 1.128$  and the supersonic and transonic flows corresponding to  $c_1 = 4.2$  and  $c_2 = 1.2205$ . In the last case, an isentropic shock at  $r = 2.15$  separates a supersonic region (for  $r < 2.15$ ) from a subsonic one (for  $r > 2.15$ ). The exact solution for the bicharacteristic variables has been imposed at all boundaries (the six sides of the computational cube) and a flow field having the exact values for  $u$  and  $a$  and zero  $v$  and  $w$  has been used as a suitable initial condition. The solution was advanced in time by means of any of the proposed methods using a constant (in time) and uniform (in space) value of  $\Delta t$ , until the average absolute value of  $\Delta C$  at all interior points was less than  $10^{-6}$ . Due to the use of the delta approach, the final steady solution is the same for all of the methods. The computed Mach number distribution along the  $x$  axis is plotted in Fig. 1 for the three flow cases versus the exact solution, for  $\Delta x = \Delta y = \Delta z = .1$ .

The solution is fairly good for the subsonic and supersonic case and qualitatively correct for the transonic one. In particular, the shock is captured in the correct mesh interval and no wiggles are present in spite of the absence of any additional dissipation. However, for shocks as strong as that given in Fig. 1, a shock-fitting procedure is warranted.

The computations were performed on a CDC Cyber 175 computer using two-point windward differences for all of the spatial derivatives.

The main purpose of this paper was to devise "efficient" implicit methods for the three-dimensional lambda-formulation equations. Therefore the performance of all of the present methods are given in Table I as the values of the  $\Delta t$  leading to the fastest convergence and the corresponding number of time steps (K) and CPU seconds.

TABLE I

Method	Subsonic Flow			Supersonic Flow			Transonic Flow		
	$\Delta t$	K	CPU	$\Delta t$	K	CPU	$\Delta t$	K	CPU
BE	.03	178	57	.03	62	21	.03	403	128
BADI	.3	35	361	.3	38	398	.3	86	887
BLGS	$\geq 5$	18	75	$\geq 2$	11	46	$\geq 10$	71	290
BPGS	$\geq 5$	29	49	$\geq 2$	11	20	$\geq 10$	76	126
SLGS	2.	22	34	.4	27	41	$\geq 10$	99	147

From Table I the following conclusions can be drawn. For the supersonic flow case the BE and BPGS methods are clearly superior; this is obvious insofar as there is no upstream propagation in the x direction and thus the implicit methods use most of the CPU time accounting for zero entries. In terms of the number of iterations, the performance of the BLGS and BPGS methods are identical as they should be (all of the x derivatives being approximated with backward differences). For the more relevant transonic and subsonic flow cases the BLGS method always requires the smallest number of iterations to converge; however, the BPGS, SLGS and BE methods are the most

efficient ones, whereas the BADI method is consistently the least competitive one. It is noteworthy that all of the Gauss-Seidel methods are very robust insofar as they maintain a quasi-optimal convergence rate over a wide range of  $\Delta t$  values. Among the three "best methods," the BPGS and the BE methods are considerably simpler to code and require less computer memory, a very critical resource when dealing with three-dimensional problems. Therefore, preliminary studies have been conducted to assess the influence on their convergence rate of the mesh size and of second-order-accurate discretization for the nonincremental terms of the governing equations. The two methods converge in a number of iterations which is roughly inversely proportional to the step size (e.g., for the subsonic flow problem convergence is reached after 261 and 52 iterations for a  $17^3$  mesh and after 309 and 59 iterations for a  $21^3$  mesh, for the BE and the BPGS methods, respectively). However, it is noteworthy that for these calculations (performed on a VAX 11/750 computer) the BE method required about 2.5 more CPU time than the BPGS method. This indicates that the solution routine for the local  $4 \times 4$  linear systems used in this study works less efficiently on the Cyber computer than the one used on the VAX and that the superiority of the BPGS method over the BE one is potentially greater than it actually appears from Table I. Also, the use of second-order-accurate differencing seems to deteriorate the convergence rate of the BPGS method less than that of the BE method. Finally, the superiority of the BPGS method (with respect to the BE method) is expected to increase even further by using a variable  $\Delta t$  [5,6,12] and when more general boundary conditions are employed; this, because the additional work will be relatively greater for the simpler BE method.

In conclusion, the BPGS method appears to be the most promising technique for solving three-dimensional compressible flow problems, by itself, or as a robust smoother within a more general multigrid procedure. However, both the BLGS and SLGS methods proposed in this study appear to be very promising alternatives to the ADI method of Refs. 5, 6 for solving two-dimensional steady flows, for which they are likely to outperform even the present BPGS method.

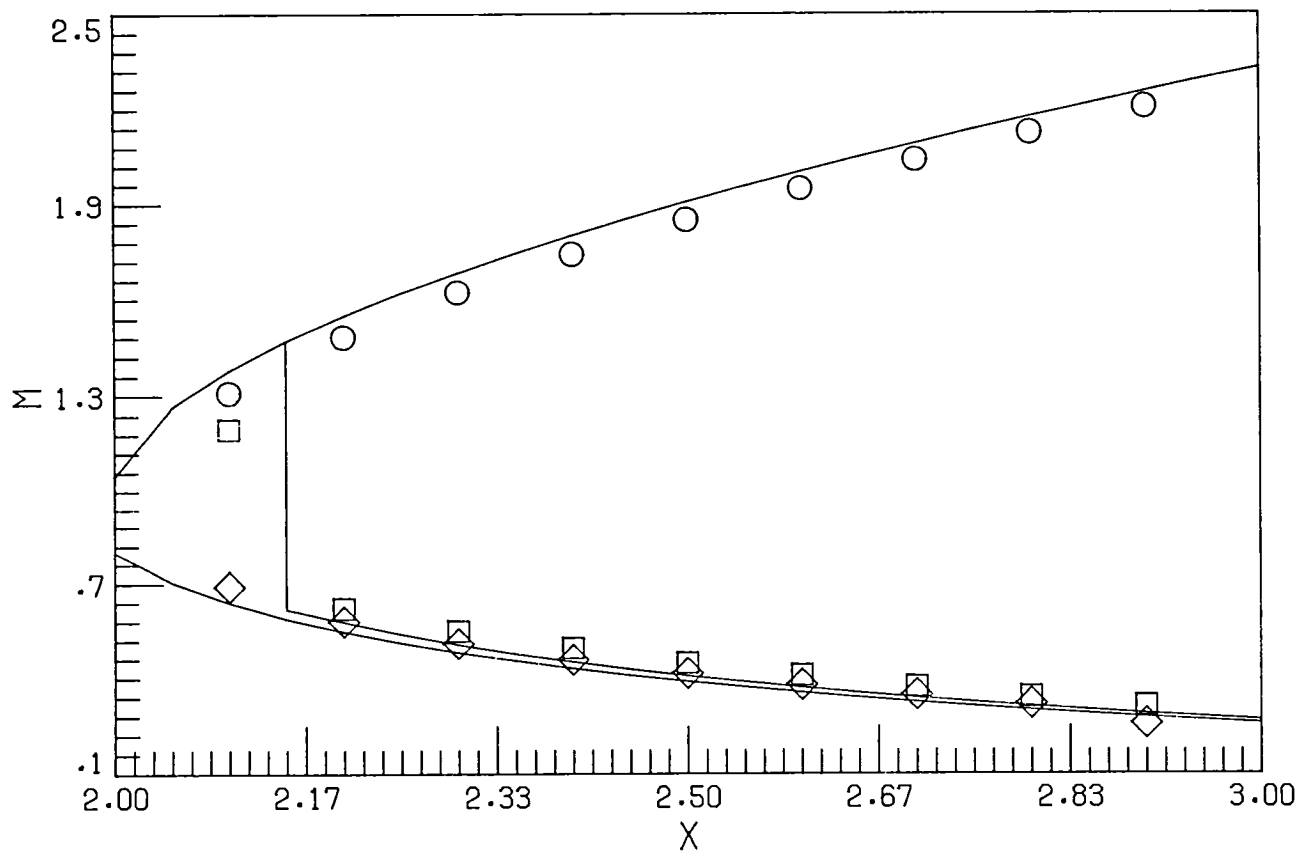
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**Figure 1.** Numerical (symbols) versus exact (solid lines) solutions for spherical source-flows.





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16. Abstract This paper derives the three-dimensional lambda-formulation equations for a general orthogonal curvilinear coordinate system and provides various block-explicit and block-implicit methods for solving them, numerically. Three model problems, characterized by subsonic, supersonic and transonic flow conditions, are used to assess the reliability and compare the efficiency of the proposed methods.					
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